Bits, Bytes and Integers

CSC 235 - Computer Organization

References

- Slides adapted from CMU

Outline

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings

Everything is bits

- Each bit is 0 or 1
- By encoding/interpreting sets of bits in various ways
  - Computers determine what to do (instructions)
  - ... and represent and manipulate numbers, sets, strings, etc.
- Why bits? Electronic implementation
  - Easy to store with bitstable elements
  - Reliably transmitted on noisy and inaccurate wires

Example: Counting in Binary

- Base 2 number representation
  - Represent 15213_{10} as 11101101101101_2
  - Represent 1.20_{10} as $1.0011001100110011[0011]\ldots_2$
  - Represent 1.5213×10^4 as $1.1101101101101_2 \times 2^{13}$

Encoding Byte Values

- Byte = 8 bits
  - Binary: 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal: 00_{16} to FF_{16}
    - Base 16 number representation
    - Use characters '0' to '9' and 'A' to 'F'
    - Typically written in most programming languages with the prefix `0x`

Encoding Byte Values

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111

Encoding Byte Values

Hex	Decimal	Binary
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Example Data Representations

C Data	Typical 32-bit	Typical-64	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
pointer	4	8	8

Boolean Algebra

- Algebraic representation of logic
  - Encode "true" as 1 and "false" as 0
  - Developed by George Boole in the 19th Century
- Operations
  - and (&): `a & b = 1` when both `a = 1` and `b = 1`
  - or (|): `a | b = 1` when either `a = 1` and `b = 1`
  - not (~): `~a = 1` when `a = 0`
  - xor (^): `a ^ b = 1` when either `a = 1` or `b = 1`, but not both

General Boolean Algebras

- Operate on Bit Vectors
  - operations applied bitwise
- Example:

$$\begin{array}{r} 01101001 \\ \&\ 01010101 \\ \hline 01000001 \end{array}$$

- All of the properties of Boolean algebra apply

Example: Representing and Manipulating Sets

- Representation
  - Width w bit vector represents subsets of $\{0, \ldots, w-1\}$
  - $a_j = 1$ if $j \in A$
- Operations
  - `&`: intersection
  - `|`: union
  - `^`: symmetric difference
  - `~`: complement

Example: Representing and Manipulating Sets

- Examples with $w = 8$
  - $x = 01101001 = \{0, 3, 5, 6\}$
  - $y = 01010101 = \{0, 2, 4, 6\}$
  - x & $y = 01000001 = \{0, 6\}$
  - x | $y = 01111101 = \{0, 2, 3, 4, 5, 6\}$

Bit-Level Operations in C

- The operations `&`, `|`, `~`, and `^` are available in C
  - apply to any "integral" data type: `long`, `int`, `short`, `char`, `unsigned`
  - arguments are viewed as bit vectors
  - arguments are applied bitwise
- Examples with `char` type
  - `~0x41` → `0xBE`
  - `~0x00` → `0xFF`
  - `0x69 & 0x55` → `0x41`

Contrast: Logical Operations in C

- The logical operations in C are `&&`, `||`, and `!`
  - zero is viewed as "false"
  - any non-zero value is viewed as "true"
  - always return 0 or 1
  - short-circuit evaluation
- Examples with `char` data type
  - `!0x41` → `0x00`
  - `!0x00` → `0x01`
  - `0x42 && 0x55` → `0x01`

Shift Operations

- Left shift: `x << y`
  - shift bit vector `x` left `y` positions
  - fill with zeros on the right
- Right shift: `x >> y`
  - shift bit vector `x` right `y` positions
  - logical shift: fill with zeros on the left
  - arithmetic shift: replicate most significant bit on the left
- Undefined behavior: shift amount less than zero or greater than bit vector length

Shift Examples

- `x = 01100010`
  - `x << 3 = 00010000`
  - logical: `x >> 2 = 00011000`
  - arithmetic: `x >> 2 = 00011000`
- `x = 10100010`
  - `x << 3 = 00010000`
  - logical: `x >> 2 = 00101000`
  - arithmetic: `x >> 2 = 11101000`

Encoding Integers

- Unsigned

$$B2U(x) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

where x is the bit vector and w is the length of the bit vector

- Signed: two's complement

$$B2T(x) = -x_{w-1} \cdot 2^{w-1} \sum_{i=0}^{w-2} x_i \cdot 2^i$$

where x is the bit vector, w is the length of the bit vector, and $-x_{x-1}$ is the sign bit

Example 3 Bit Integer Encodings

value	unsigned	two's complement
000	(0+0+0) = 0	(0+0+0) = 0
001	(0+0+1) = 1	(0+0+1) = 1
010	(0+2+0) = 2	(0+2+0) = 2
011	(0+2+1) = 3	(0+2+1) = 3
100	(4+0+0) = 4	(−4+0+0) = −4
101	(4+0+1) = 5	(−4+0+1) = −3
110	(4+2+0) = 6	(−4+2+0) = −2
111	(4+2+1) = 7	(−4+2+1) = −1

Numeric Ranges

- Unsigned values
  - min $= 0$
  - max $= 2^w - 1$
- Two's complement values
  - min $= -2^{w-1}$
  - max $= 2^{w-1} - 1$

Example Numeric Ranges

- Values where $w = 16$

	decimal	hex	binary
unsigned max	65535	FF FF	11111111 11111111
signed max	32767	7F FF	01111111 11111111
signed min	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

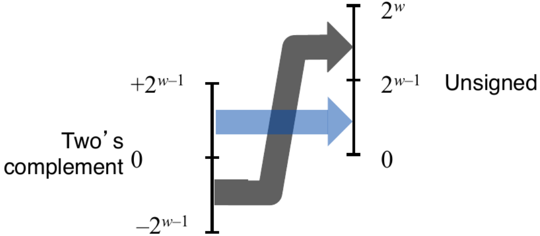
Unsigned and Signed Numeric Values

- Equivalence
  - Same encodings for non-negative values
- Uniqueness
  - Every bit pattern represents a unique integer value
  - Each representable integer has a unique bit encoding
- Can invert mappings
  - unsigned bit pattern $= U2B(x) = B2U^{-1}(x)$
  - two's complement bit pattern $= T2B(x) = B2T^{-1}(x)$

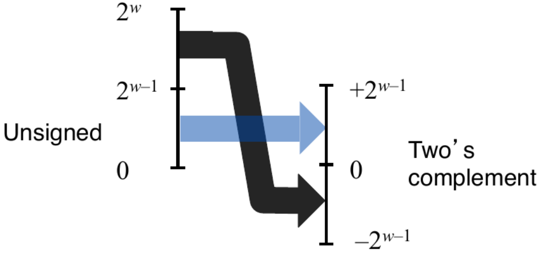
Mapping Between Signed and Unsigned

- Mappings between unsigned and two's complement numbers: keep the bit representation and reinterpret.
- Two's complement to unsigned: $T2B \circ B2U$
- Unsigned to two's complement: $U2B \circ B2T$

Signed to Unsigned



Unsigned to Signed



Signed vs. Unsigned in C

- Constants
  - By default are considered to be signed integers
  - Unsigned if the suffix is "U", for example `42U`
- Casting
  - Explicit casting between signed and unsigned same as $U2T$ and $T2U$
  - Implicit casting also occurs via assignments and procedure calls

Casting Surprises

- Expression evaluation
  - If there is a mix of unsigned and signed integers in a single expression, then signed values are implicilty cast to unsigned values.
  - Including comparison operations: `<`, `>`, `==`, `<=`, `>=`
- Examples

Operand 1	Operand 2	Relation	Evaluation
0	0U	$==$	unsigned
-1	0	$<$	signed
-1	0U	$>$	unsigned
-1	-2	$>$	signed

Unsigned vs. Signed in C

- Easy to make mistakes
- Example 1

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
    a[i] += a[i+1]
```

- Example 2

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i -= DELTA)
    ...
```

Summary: Casting Rules

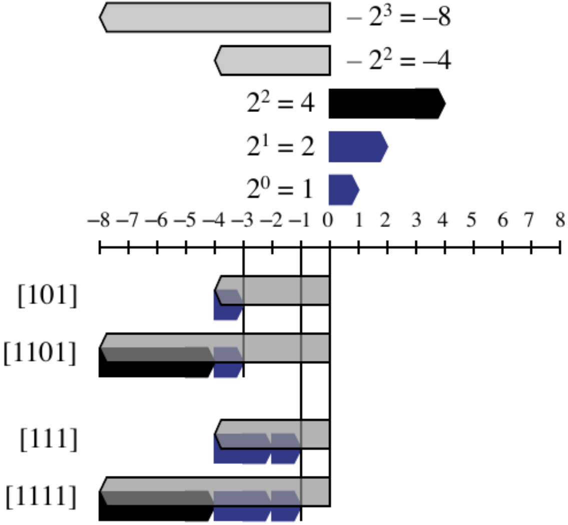
- Bit pattern is maintained, but reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- An expression containing signed and unsigned `ints` implicitly casts the signed `ints` to unsigned `ints`

Sign Extension

- Task
  - Given w-bit signed integer x
  - Convert it to $w + k$ bit integer x' with the same value
- Rule
  - Make k copies of the sign bit:
  - $x' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_0$
- C automatically performs sign extension

Sign Extension Example

- Example of sign extensions from $w = 3$ to $w = 4$



Truncation

- Task:
  - Given $k + w$-bit signed or unsigned integer x
  - Convert it to w-bit integer x' with the same value for "small enough" x
- Rule:
  - Drop top k bits:
  - $x' = x_{w-1}, x_{w-2}, \ldots, x_0$

Summary: Expanding and Truncating Rules

- Expanding (e.g. `short` to `int`)
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result
- Truncating (e.g. `int` to `short`)
  - Unsigned/signed: bits are truncated
  - Result is reinterpreted
  - Unsigned: modulus operation
  - Signed: similar to modulus
  - For small (in magnitude) numbers yields expected behavior

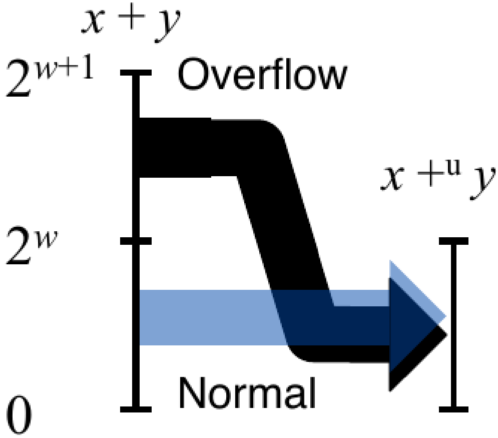
Unsigned Addition

- $UAdd_w(u, v)$
  - Operands: w bits
  - True sum: $w + 1$ bits
  - Discard carry: w bits
- Standard addition function ignores carry output
- Implements modular arithmetic

$$s = UAdd_w(u, v) = u + v \texttt{ mod } 2^w$$

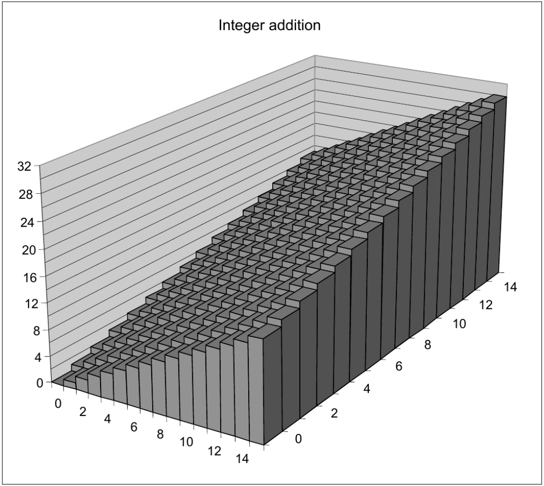
UAdd Overflow

- Implements modular arithmetic
  $s = UAdd_w(u, v) = u + v \bmod 2^w$



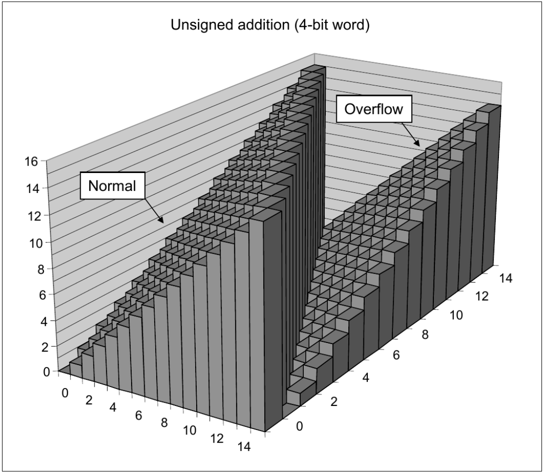
Visualizing Mathematical Integer Addition

- $Add_4(u, v)$



Visualizing Unsigned Integer Addition

- $UAdd_4(u, v)$

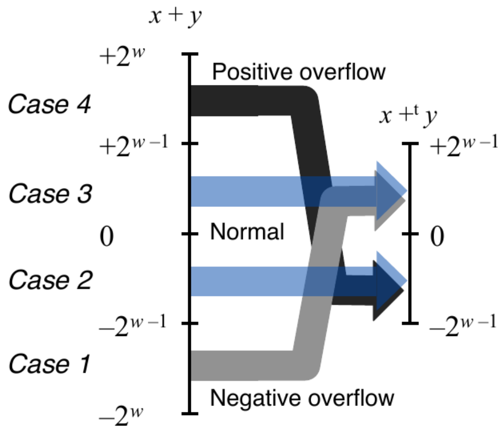


Two's Complement Addition

- $TAdd_w(u, v)$
  - Operands: w bits
  - True sum: $w + 1$ bits
  - Discard carry: w bits
- *TAdd* and *Uadd* have identical bit level behavior

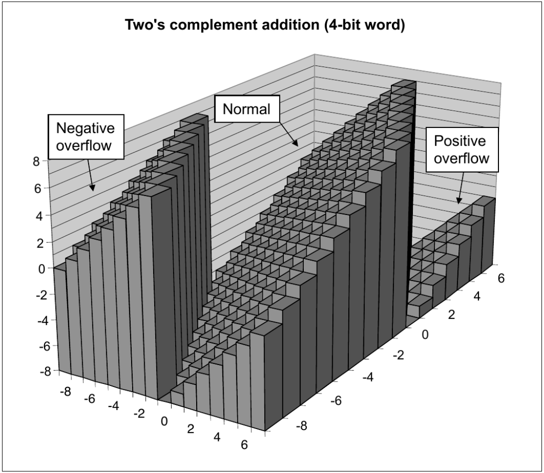
TAdd Overflow

- True add requires $w + 1$ bits; drop off the most significant bit and interpret as 2's complement integer



Visualizing Two's Complement Addition

- $TAdd_4(u, v)$



Integer Multiplication

- Problem: the exact product of w-bit numbers u, v might have a result that exceeds w bits.
  - Unsigned: up to $2w$ bits
  - Two's complement min (negative): up to $2w - 1$ bits
  - Two's complement max (positive): up to $2w$ bits
- Maintaining exact results
  - would need to keep expanding word size with each product computed
  - is done in software if needed

Unsigned Multiplication in C

- $UMul_w(u, v)$
  - Operands: w bits
  - True product: $2w$ bits
  - Discard w bits: w bits
- Implements modular arithmetic

$$s = UMul_w(u, v) = u + v \bmod 2^w$$

Signed Multiplication in C

- $TMul_w(u, v)$
  - Operands: w bits
  - True product: $2w$ bits
  - Discard w bits: w bits
- Ignores high order w bits, some of which are different for signed vs. unsigned multiplication

Power-of-2 Multiply with Shift

- Operation `u << k`
  - Gives $u \cdot 2^k$ for both signed and unsigned
  - Operands: w bits
  - True product $w + k$ bits
  - Discard k bits: w bits

Unsigned Power-of-2 Divide with Shift

- Operation `u >> k`
  - Gives

$$\left\lfloor \frac{u}{2^k} \right\rfloor$$

  - Uses logical shift

Signed Power-of-2 Divide with Shift

- Operation `u >> k`
  - Gives

$$\left\lfloor \frac{u}{2^k} \right\rfloor$$

  - Uses arithmetic shift
  - Rounds wrong direction when $u < 0$

Correct Signed Power-of-2 Divide with Shift

- Quotient of negative number power of 2
  - Want

$$\left\lceil \frac{u}{2^k} \right\rceil$$

  - Compute as

$$\left\lfloor \frac{u + 2^k - 1}{2^k} \right\rfloor$$

    - In C: `(u + (1<<k) - 1) >> k`
    - Biases dividend toward 0

Negation: Complement and Increment

- Negate through complement and increment

  `~x + 1 = -x`

- Examples

Value	x	~x	~x+1	Result
15213	3B6D	C492	C493	-15213
0	0000	FFFF	0000	0
TMin	8000	7FFF	8000	TMin

Arithmetic: Basic Rules

- Addition
  - Unsigned/signed: normal addition followed by truncate
  - Unsigned: addition $\texttt{mod}\ 2^w$
  - Signed: modified addition $\texttt{mod}\ 2^w$ (result in proper range)
- Multiplication
  - Unsigned/signed: normal multiplication followed by truncate
  - Unsigned: multiplication $\texttt{mod}\ 2^w$
  - Signed: modified multiplication $\texttt{mod}\ 2^w$ (result in proper range)

Byte-Oriented Memory Organization

- Programs refer to data by address
  - Conceptually envision it as a very large array of bytes
  - An address is like an index into that array, and a pointer variable stores an address
- Note: system provides private address space to each "process"
  - Think of a process as a program being executed
  - So, a program can clobber its own data, but not that of others

Machine Words

- Any given computer has a “word size”
  - Nominal size of integer-valued data
- Until recently, most machines used 32 bits (4 bytes) as a word size
- Increasingly, machines have 64 bit word size
- Machines still support multiple data formats
  - Fractions or multiples of word size
  - Always integral number of bytes

Word-Oriented Memory Organization

- Addresses specify byte locations
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32 bit) or 8 (64 bit)

Byte Ordering

- How are the bytes within a multi-byte word ordered in memory?
- Conventions
  - Big endian: least significant byte has highest address
  - Little endian: least significant byte has lowest address
- Example: 4-byte value of `0x1234567`
  - Big endian: `01 23 45 67`
  - Little endian: `67 45 23 01`

Examining Data Representations

- Code to print byte representation of data

```
typedef unsigned char *pointer;
void show_bytes(pointer start, size_t len) {
  size_t i;
  for (i = 0; i < len; i++) {
      printf("%p\t0x%.2x\n", start+i, start[i]);
  }
  printf("\n");
}
```

Representing Strings

- Strings in C
  - Represented by an array of characters
  - Each character is encoded in ASCII format
  - Strings should be null terminated (final character = 0)
- Compatibility
  - Byte ordering is not an issue

Reading Byte-Reversed Listings

- Disassembly
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- Example Fragment

```
Address     Instruction code      Assembly Rendition
8048365:    5b                    pop
8048366:    81 c3 ab 12 00 00     add     $0x12ab,%ebx
804836c:    83 bb 28 00 00 00 00  cmpl    $0x0,0x28(%ebx)
```

Summary

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- Representations in memory, pointers, strings